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"Free-Space" Boundary Conditions for the Time Dependent Wave Equation

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Boundary conditions for the discrete wave equation which act like an infinite region of free space in contact with the computational region can be constructed using projection operators. Propagating and evanescent waves coming from within the computational region generate no reflected waves as they cross the boundary. At the same time arbitrary waves may be launched into the computational region. Well-known projection operators for one-dimensional waves may be used for this purpose in one dimension. Extensions of these operators to higher dimensions along with numerically efficient approximations to them are described for higher-dimensional problems. The separation of waves into ingoing and outgoing waves inherent in these boundary conditions greatly facilitates diagnostics.

I. INTRODUCTION

A large class of computational problems exists in which reflection of waves back into the computational region by the boundaries is undesirable. In many of these the simultaneous launching of known waves into the computational region is also required. Since computer restrictions put a limit on the size of a region which can be handled, placing the boundaries a long way from the region of computational interest is impractical. Thus the need for boundary conditions which act as an infinite region of free space is apparent.

The primary requirement of such a boundary is the ability to absorb waves incident upon it rather than reflect them. A narrow region in which a dissipation is added to the wave equation is an obvious possible approach. Thus waves impinging on such a boundary are damped on the way in, reflected by a conventional boundary condition, and damped further on the way out. This technique, though not altogether useless, suffers from the following disadvantage. To get substantial absorption over a wide range of angles and wavelengths, the dissipation must occupy a region which is several of the largest wavelengths in width. The region is thus not particularly narrow and is, of course, not available for computing the problem of interest.

A similar approach using a system of dashpots has been formulated by Lysmer and Kuhlemeyer [1]. For a narrow range of frequencies, and hence wavelength, they obtain reasonable results for a wide range of angles. An interesting approach involving the propagation of D'Alembert forces has been considered by Ang and Newmark [2]. A method which is exact to roundoff has been discussed by Smith [3]. This method requires the integration over the full computational mesh of 2^n wave equations where n is the number of boundaries at which zero-reflection is desired. It is thus computationally inefficient. Elvius and Sundstrom [4] discuss a method in which information about the characteristics of the wave equation of interest is used to construct an absorbing boundary. Their results are inconclusive and they suggest that further work is required. Approximate methods based on extrapolation are discussed by Chen [5].

The work presented here is based on projection operators. These can be defined exactly for the problems of interest, but, in the interest of numerical efficiency, approximations are introduced. Early unpublished work by Freidberg and Morse led to an extremely efficient method for the one-dimensional problem. No processing of past data was required and, of course, the projection operator methods only use data at the boundary. An improved version of their method is briefly discussed in Section III. C. W. Nielson's unpublished extension of this work to the absorption of all waves traveling at a fixed predetermined angle to the boundary in two- and three-dimensional problems is also mentioned.

II. Analysis of Boundary Conditions

Consider the simplest differencing of the two-dimensional wave equations

$$(\Delta_x^2/D_x^2)\,A + (\Delta_y^2/D_y^2)\,A = (\Delta_t^2/c^2\tau^2)\,A, \tag{1}$$

where Δ_x^2 is the second-central difference in x, etc., D_x and D_y are the mesh spacings in x and y, and τ is the time step. $A = A^j_{l,m}$ is defined only at discrete points in space $x = lD_x$ and $y = mD_y$ and discrete points in time $t = j\tau$. This difference equation has plane wave solutions of the form

$$A^j_{l,m} = A \exp i[k_x l D_x + k_y m D_y - \omega j \tau], \tag{2}$$

where ω, k_x, and k_y satisfy the dispersion relation

$$\sin^2(\tfrac{1}{2}\omega\tau) = (c\tau/D_x)^2 \sin^2(\tfrac{1}{2}k_x D_x) + (c\tau/D_y)^2 \sin^2(\tfrac{1}{2}k_y D_y). \tag{3}$$

If a plane wave such as this impinges on a fixed boundary at $x = x_{\max}$, the additional function required to satisfy the boundary condition must have the same frequency, ω, and wavenumber, k_y, along the boundary. Since the dispersion relation is symmetric in k_x, only waves with $k_x = -k_x$ can be used to satisfy the boundary condition. Most common boundary conditions require the presence of a wave with $k_x = -k_x$ in addition to the incident wave to be satisfied and, hence, reflection is generated.

A general disturbance in the medium can, of course, be represented as a sum or integral (depending on the boundary conditions) over the k_x's, k_y's, and ω's subject to the restriction that the dispersion relation be satisfied for the individual components. Thus if a boundary condition is constructed that is a linear operation and is satisfied by all outgoing waves ($\omega/k_x > 0$) with values of ω, k_x, and k_y consistent with the dispersion relation, it will absorb a general disturbance incident upon it.

From the above discussion it is clear that the analysis of the reflection generated by various boundary conditions is best carried out in Fourier space (ω, k_x, and k_y space). The boundary conditions to be discussed are themselves most simply formulated in Fourier space. Their complicated operator character arises only upon translating the simple multiplicative functions of ω and k_y which are required back into y and t space for use in time dependent simulation problems.

A boundary condition for a difference equation which contains second order differences in space is a relation between the value of the function and its first normal derivative. This relation may include other functions of time, of course. In a time dependent problem past values of the function in the computational region and at the boundary are in principle available and can be included in the above additional functions of time along with information regarding waves to be launched into the medium. The best boundary condition uses a minimum of past data stored in a form which minimizes the additional storage and the additional operations required to process these data. The boundary conditions to be described below require past data only at the boundary and the updating of three to six functions of these data which are then combined with the information regarding waves to be launched at the boundary. They are, thus, quite efficient and are capable of generating reflection coefficients less than 1 % over a wide range of frequencies and angles.

III. Projection Operators in Boundary Conditions

Consider the boundary condition

$$\Delta_t \sigma_x A^j_{l,m} \pm \frac{c\tau}{D_x} \hat{G} \Delta_x \sigma_t A^j_{l,m} = 4\Delta_t S_m{}^j. \tag{4}$$

Δ_t and Δ_x are forward differences, while σ_x and σ_t are forward sums (e.g. $\sigma_x A^j_{l,m} = A^j_{l,m} + A^j_{l+1,m}$) introduced to obtain centering. With suitably chosen operator, $\hat{G}$, the LHS of Eq. (4) is a projection operation which projects out left-going waves only with the plus sign and right-going waves only with the minus sign.

The RHS is a source function which launches waves back into the computational region if it is nonzero. Consider a solution of the wave equation which consists of a wave propagating to the right with amplitude R, and a wave propagating to the left with amplitude L, and a source function with the same y and t dependence.

$$\begin{aligned} A^j_{l,m} &= [R \exp(ik_x l D_x) + L \exp(-ik_x l D_x)] \exp i(k_y m D_y - \omega j \tau), \\ S_m{}^j &= S \exp i(k_y m D_y - \omega j \tau). \end{aligned} \tag{5}$$

Substituting Eqs. (5) into Eq. (4) with the plus sign leads to an expression for the amplitude of the left-going wave in terms of the amplitude of the right-going wave and the source.

$$\begin{aligned} L \exp[-ik_x(l + \tfrac{1}{2}) D_x] = {} & [(G - G_0)/(G + G_0)] R \exp[ik_x(l + \tfrac{1}{2}) D_x] \\ & + [2G_0 \sec(\tfrac{1}{2}k_x D_x)/(G + G_0)] S, \end{aligned} \tag{6}$$

where

$$G_0 = (D_x/c\tau)[\tan(\tfrac{1}{2}\omega\tau)/\tan(\tfrac{1}{2}k_x D_x)] \tag{7}$$

and G is $\hat{G}$ in $(\omega, \mathbf{k})$-space. Setting $S = 0$, and ignoring the factors which relate the complex L and R to their values at $x = 0$, gives the reflection coefficient which depends on G.

$$R_G = L/R = (G - G_0)/(G + G_0). \tag{8}$$

If $G = G_0$ the reflection coefficient is zero. Thus the problem reduces to finding numerically useful approximations to the operator G_0.

It is instructive to consider the limiting case τ, D_x, $D_y \to 0$. In this limit $G_0 = \omega/ck_x = \sec\theta$, where θ is the angle between $\mathbf{k}$ and the x-direction for propagating waves. In one-dimension, $G_0 = 1$ and its operator character is important only for waves which are being treated unphysically anyway. A boundary condition similar to Eq. (4) with $G = 1$ was obtained some years ago by Freidberg and Morse at Los Alamos and has been used extensively in one-dimensional codes since that time. C. W. Nielson further showed that an extension of this one-dimensional boundary condition, namely $G = \sec\theta_0$, could be used in two and three dimensions to absorb all waves traveling at the fixed angle, θ_0, with respect to the x-axis.

The author was the first to formulate the problem as described above and construct a numerically useful operator for G which would absorb waves at all angles in a two-dimensional system [6]. An approximate form for G_0^{-1} was obtained similar to the one to be described below. Coefficients were evaluated by minimizing the reflection coefficient over a range of angles from $\theta = 1$ to 89° such that $R_G < .01$ over the range. Further details can be found in Ref. [6].

This boundary condition has been superceded by the two described below. It was found that equally good absorption could be obtained by approximating G_0 instead of G_0^{-1} and in doing so the time derivatives of the ingoing and outgoing waves were directly available for diagnostics rather than the normal derivatives. It was also found more convenient to supply the time derivatives of the ingoing waves as in Eq. (4) rather than the normal derivative which was required by the boundary condition in Ref. [6].

IV. Boundary Condition for Traveling Waves Only

A useful approximation to G_0 for cases in which all waves to be encountered are traveling waves can be obtained as follows. Note first that

$$G_0 = \omega/ck_x = [1 - (c^2k_y^2/\omega^2)]^{-1/2} \tag{9}$$

in the limit D_x, D_y, $\tau \to 0$. G_0 can be written as an operation in space and time in the following way.

$$\hat{G}_0 \cong [1 - (c\tau/D_y)^2(\Delta_y^2/\Delta_t^2)]^{-1/2}. \tag{10}$$

The simple power series expansion suggests itself as a first approach, but, aside from the fact that it converges slowly, it is unstable in time [6]. Thus it is numerically useless. The following expression, however, is stable and leads to an excellent approximation for $ck_y/\omega < 1$.

$$\hat{G} = 1 + \sum_{n=1}^{N} \hat{g}_n, \tag{11}$$

where

$$\hat{g}_n = \frac{\alpha_n(c\tau/D_y)^2\,(\Delta_y^2/\Delta_t^2)}{1 - \beta_n(c\tau/D_y)^2\,(\Delta_y^2/\Delta_t^2)}. \tag{12}$$

Substituting this for $\hat{G}$ into Eq. (4) leads to the following expression for the boundary condition.

$$\Delta_t\sigma_x A_{l,m}^j \pm \frac{c\tau}{D_x}\Delta_x\sigma_t A_{l,m}^j = 4\Delta_t S_m^j \mp \frac{c\tau}{D_x}\sum_{n=1}^{N} h_{n,m}^{j+1/2}, \tag{13}$$

$$h_{n,m}^{j+1/2} = \hat{g}_n\Delta_x\sigma_t A_{l,m}^j. \tag{14}$$

The meaning of Eq. (14) is made clear by rewritting it as follows.

$$[(\Delta_t^2/c^2\tau^2) - \beta_n(\Delta_y^2/D_y^2)]\, h_{n,m}^{j+1/2} = \alpha_n(\Delta_y^2/D_y^2)\, \Delta_x\sigma_t A_{l,m}^j\,. \tag{15}$$

Thus the h's are correction functions to the one-dimensional boundary condition which are functions of past data at the boundary. As the problem proceeds in time these quantities must be updated using Eq. (15). Since they are not functions of the A's at the point in time at which the boundary condition is applied, they are updated first. Equation (13) is then solved implicity for the "ghost" cell values needed to apply the boundary condition.

Having obtained an explicit statement of the boundary condition, Eqs. (13)–(15), one can analyze it exactly in Fourier space as previously described. Reflection coefficients as a function of k_y and ω, or in terms of angle of propagation measured from the normal can then be calculated. In order to obtain reflection coefficients less than 0.01 for a range of angles from 0° to 89°, $N = 3$ terms in $\hat{G}$ are required. The values of α_n and β_n are then obtained by mimimizing the following function.

$$F = \sum_{\theta=1^\circ}^{89^\circ} W(\theta)[R_G(\alpha_n, \beta_n, \theta)]^2.$$

Best results were obtained using as the weighting function $W(\theta) = (\cos\theta)^{1/2}$.

For a set of α's and β's obtained in this manner the resulting reflection coefficient

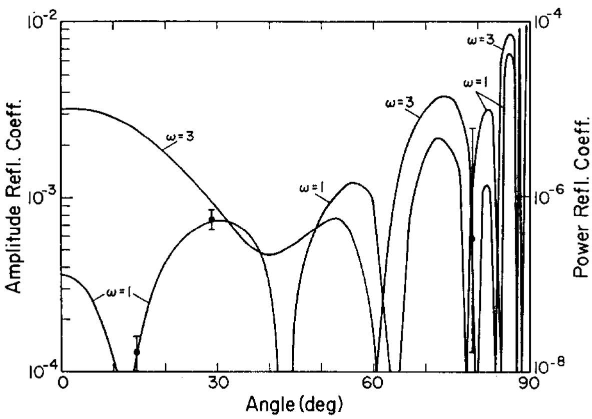

FIG. 1. The reflection coefficient vs angle of propagation measured from normal is presented for the boundary condition described by Eqs. (13)–(15) with $N = 3$. The three values of α used were 0.3264, 0.1272, and 0.0309, and the corresponding values of β were 0.7375, 0.98384, and 0.9996472 respectively. Other parameters were $c = 1$, $\tau = 0.0654$, and $D_x = D_y = 0.1$. The data points are measured values of the reflection coefficient for $\omega = 1$.

as a function of θ is shown in Fig. 1. The reflection coefficient rises as the frequency increases and numerical dispersion increasingly affects the waves. In cases where these waves must be absorbed with a high degree of accuracy the form of $\hat{G}$ must be suitably modified. This form of $\hat{G}$, however, is quite adequate for waves in the physically accurate regime even for other methods of differencing the wave equation. It should also be noted that the values of the β_n's are all less than one. Hence the Courant conditions for the boundary functions are less restrictive than that for the interior.

The data points shown Fig. 1 are measured values of the reflection coefficient. Equation (1) was solved on a two-dimensional mesh using periodic boundary conditions in y. Equations (13)–(15) were used for the boundary conditions in x. The plus sign was used at the right boundary, while the minus sign was used at the left. The problem was initialized with A and its time derivative everywhere equal to zero. The boundary correction functions were also initialized with their values and time derivatives equal to zero. The source at the right boundary was zero throughout, while at the left boundary a wave was launched with a fixed k_y as follows.

$$\frac{\Delta_t}{\tau} S_m{}^j = f_p\left(\frac{j\tau}{T}\right) \sin(j\tau - k_y m D_y),$$

$$f_p(s) = \begin{cases} \dfrac{\Gamma(2p)}{[\Gamma(p)]^2} \displaystyle\int_0^s [u(1-u)]^{p-1}\, du, & 0 \leqslant s \leqslant 1, \\ 1, & s > 1. \end{cases}$$

$T = 30$ with $p = 2$ was used for the points at $\theta = 14°$ and $29°$, while $T = 500$ with $p = 4$ was used at $\theta = 79°$.

To measure the amplitude of the reflected wave returning to the left boundary, the projection operator for left-traveling waves was applied at the left boundary. After evaluating the terms on the LHS of Eq. (4) by requiring that their difference be determined by the source, i.e. by applying the boundary condition, their values may be combined with the plus sign to obtain information about the reflected wave. In most cases of practical interest the amplitude of the returning wave is a sizeable fraction of the launched wave; the errors in the projection operator are negligible; and the desired information is obtained directly. In this case, however, the errors in the projection operator operating on the launched wave are the same order of magnitude as the amplitude of the returning wave. Hence it is necessary to obtain an expression for the output of the projection operator in terms of the errors in the projection operators at both boundaries. The contributions from both boundaries to this expression can be written in terms of the reflection coefficients at the two boundaries which are the same. This expression is then solved for the reflection coefficient in terms of the output of the projection operator, the k_x of the wave in the intervening region and separation between the boundaries. The resulting values of the reflection coefficient are plotted in Fig. 1.

The error bars reflect, primarily, the effects of the finite turn-on time of the wave. The resulting frequency spread translates into a spread in angles, $\delta\theta$, as follows.

$$\delta\theta \simeq -\tan\theta(\delta\omega/\omega).$$

For the points at 14° and 29°, the angular spread occupies a range of angles which is handled well by the boundary condition. Hence the error bars are small. At 79°, however, a small but nonnegligible fraction of the energy was radiated at 90° and beyond, i.e. in the evanescent region where the boundary condition is admittedly incorrect. Undamped resonances in the boundary whose energy cannot be radiated away because their frequencies correspond to evanescent waves were driven up. These oscillations are superimposed on the oscillations of interest and can only be separated out by Fourier analyzing over many oscillations, which was not done. When this case was done with a $p = 2$, $T = 30$ turn-on, a sizeable fraction, 10% to 20%, of the energy was radiated in the evanescent regime, and the reflection coefficient was effectively unmeasurable. It is therefore concluded that this boundary condition is useful for angles up to 60° or 70° when reasonable amplitude fluctuations in time are assumed, i.e., $\delta\omega/\omega \lesssim 0.1$.

V. Boundary Condition for Traveling and Evanescent Waves

In some problems evanescent waves having $ck_y/\omega > 1$ arise. These waves will not be handled correctly by the boundary condition described in Section IV. Boundary conditions that handle these waves with the same degree of accuracy while simultaneously handling propagating waves as before, however, can be constructed. Approximately twice as much additional storage is required as well as a factor of more than 2 in time to update the auxiliary quantities.

The expressions of interest for evanescent waves, $ck_y/\omega > 1$, may be obtained from Eqs. (2)–(8) by replacing k_x with iK_x, $K_x > 0$. Right-traveling waves become exponentially decreasing to the right consistent with their sources' being located in the computational region, while left-traveling waves become exponentially decreasing away from the boundary consistent with the boundary's being their source. In Fourier space G_0 may now be defined over the range $0 \leqslant |ck_y/\omega| \leqslant \infty$ as follows.

$$G_0 = \begin{cases} (D_x/c\tau)[\tan(\frac{1}{2}\omega\tau)/\tan(\frac{1}{2}k_xD_x)], & k_x^2 > 0, \\ (D_x/c\tau)[\tan(\frac{1}{2}\omega\tau)/i\tanh(\frac{1}{2}K_xD_x)], & K_x^2 > 0. \end{cases} \quad (16)$$

The problem thus reduces to the search for numerically useful approximations to G_0 over the extended range.

In order to motivate the choice of G, consider the expression G_0 in the evanescent region for D_x, D_y, $\tau \to 0$. In this limit, the dispersion relation may be combined with Eq. (16) to obtain the following expression.

$$G_0 = \frac{-i\omega}{cK_x} = \frac{1}{[1 - (\omega^2/c^2k_y^2)]^{1/2}} \frac{(-i\omega)}{c \mid k_y \mid}. \tag{17}$$

Introducing an approximate expression for the reciprocal square root similar to that used before, one obtains

$$\begin{aligned} G &= \sum_{n=1}^{N} \frac{\tilde{\alpha}_n}{1 - \tilde{\beta}_n(\omega^2/c^2k_y^2)} \frac{(-i\omega)}{c \mid k_y \mid} \\ &= \sum_{n=1}^{N} \frac{\tilde{\alpha}_n \tilde{\beta}_n^{-1}}{1 - \tilde{\beta}_n^{-1}(c^2k_y^2/\omega^2)} \frac{c \mid k_y \mid}{(-i\omega)}. \end{aligned} \tag{18}$$

If $-i\omega$ is replaced by Δ_t/τ, the resultant operator involves integrations in time only. Hence only past data are required to evaluate it. Along with the previously encountered derivatives in y obtained when $-k_y^2$ is replaced by $D_y^{-2}\Delta_y^2$, this expression involves the operation $\mid \Delta_y \mid$. If one chooses to work with boundary data Fourier analyzed in y, this operation is simply multiplication of each Fourier component by the appropriate constant, $\mid 2 \sin(\frac{1}{2}k_y D_y) \mid$. When the boundary conditions in y allow a finite Fourier series, working with the Fourier components in this manner appears to be the best procedure. When such a Fourier series is not allowed, this operator may be defined as follows. Consider an arbitrary function of y; Fourier transform in y; replace each Fourier component by its value multiplied by $\mid 2 \sin(\frac{1}{2}k_y D_y) \mid$; and then perform the inverse. This procedure leads to the following "integral" expression for $\mid \Delta_y \mid$ when a finite Fourier series is allowed.

$$\mid \Delta_y \mid f(mD_y) = \sum_{m'=1-M/2}^{M/2} \eta(m, m') f(m'D_y), \tag{19}$$

where

$$\eta(m, m') = (2/M)\{\sin(\pi/M)/\cos[2\pi(m - m')/M] - \cos[\pi/M]\}.$$

In the limit $M \to \infty$, the following expression for η is obtained which is the desired result.

$$\eta(m, m') = (4/\pi)\{1/[1 - 4(m - m')^2]\}. \tag{20}$$

The final realization of this operator on a finite domain will then depend on the choice of boundary conditions in y.

One can proceed in this manner and obtain an explicit form for a boundary condition which handles evanescent waves correctly but does not handle propagating waves correctly. It will have undamped resonances in the propagating region. Moreover the Courant condition on the boundary functions will be more restrictive than the Courant condition for the interior. It is, therefore, probably not of general interest.

To obtain an expression for G in the more general case, it is reasonable to attempt to combine expressions valid in the two regions in some way. Combining expressions obtained from Eqs. (11)–(13) in the limit D_x, D_y, $\tau \to 0$ with Eq. (18) the following expression is obtained.

$$G = 1 + \sum_{n=1}^{N} \left[\frac{\alpha_n(c^2k_y{}^2/\omega^2)}{1 - \beta_n(c^2k_y{}^2/\omega^2)} + \frac{\tilde{\alpha}_n\tilde{\beta}_n^{-1}}{1 - \tilde{\beta}_n^{-1}(c^2k_y{}^2/\omega^2)} \frac{c\mid k_y \mid}{(-i\omega)} \right]. \tag{21}$$

This expression is inadequate, of course, because each term is nonzero in the region where the other by itself is accurate. In the propagating region, $ck_y/\omega < 1$, G should be purely real, but an imaginary part is introduced by the expression required in the evanescent region. Similarly in the evanescent region G should be purely imaginary but a real part is introduced by the expression required in the propagating region. It can be shown, however, that the inclusion of damping terms in the "wave equations" in the denominators leads to terms which can effectively cancel these unwanted contributions in both regions. Furthermore, when the reflection coefficient that is obtained from such an expansion is minimized to obtain the optimum values of the α's, $\tilde{\alpha}$'s, β's, $\tilde{\beta}$'s, and the newly added γ's and $\tilde{\gamma}$'s it is found that $\beta_n = \tilde{\beta}_n = 1$, $\alpha_n = \tilde{\alpha}_n$, and $\gamma_n = \tilde{\gamma}_n$ for all n.

For these reasons the following choice is made.

$$\hat{G} = \sum_{n=1}^{N} \alpha_n \left[1 + \frac{(c\tau/D_y)^2\,(\Delta_y{}^2/\Delta_t{}^2) + (2/\sigma_t)(1-\gamma_n)(c\tau/D_y)(\mid \Delta_y \mid/\Delta_t)}{1 + (\sigma_t/2)\,\gamma_n(c\tau/D_y)(\mid \Delta_y \mid/\Delta_t) - (c^2\tau^2/D_y{}^2)(\Delta_y{}^2/\Delta_t{}^2)}\right], \tag{22}$$

with the restrictions $\sum_{n=1}^{N} \alpha_n = 1$, $\sum_{n=1}^{N} \alpha_n\gamma_n = 1$. The σ_t's are backward or forward summation operators used to obtain centering in the final equations. In a code in which boundary conditions in y allow a discrete Fourier analysis it is most reasonable to work with Fourier components in y since they will probably be desired for diagnostic purpose anyway.

Assuming that the fields have been Fourier analyzed in y, m is now the Fourier component index, and $F_m = (2c\tau/D_y)\mid \sin(\pi m/M)\mid$, Eq. (22) can be rewritten in the following form.

$$\hat{G} = \sum_{n=1}^{N} \alpha_n \left[1 + \frac{(-F_m^2/\Delta_t^2) + (2/\sigma_t)(1 - \gamma_n)(F_m/\Delta_t)}{1 + (\sigma_t/2)\,\gamma_n(F_m/\Delta_t) + (F_m^2/\Delta_t^2)}\right]. \tag{23}$$

Substituting this expression for G into Eq. (4) gives the boundary condition

$$\Delta_t \sigma_x A_{l,m}^j \pm \frac{c\tau}{D_x}\left(\sum_{n=1}^{N} \alpha_n\right) \Delta_x \sigma_t A_{l,m}^j = 4\Delta_t S_m^{\ j} \mp \frac{c\tau}{D_x} \sum_{n=1}^{N} h_{n,m}^{j+1/2}. \tag{24}$$

And the correction functions, h, must satisfy the equation

$$(\Delta_t^2 + \tfrac{1}{2}\gamma_n F_m \sigma_t \Delta_t + F_m^2)\, h_{n,m}^{j+1/2} = \alpha_n[-F_m^{\ 2}\sigma_t + 2(1 - \gamma_n)\, F_m \Delta_t]\, \Delta_x A_{l,m}^j\,. \tag{25}$$

On the LHS of Eq. (25) the $\frac{1}{2}\sigma_t \Delta_t$ indicates an average time difference centered on $j + \frac{1}{2}$. This choice of time differencing makes the Courant condition on the boundary functions less restrictive than the Courant condition for the interior. The correction functions, h, play the same role as those discussed in Section IV, and the procedure for applying the boundary condition is identical to the procedure presented in Section IV.

By Fourier analyzing Eqs. (24)–(25) an explicit expression for the reflection coefficient as a function of $\sin(u) = (c\tau/D_y)\sin(\frac{1}{2}k_y D_y)/\sin(\frac{1}{2}\omega\tau)$ can be obtained. By minimizing the square of the magnitude of the reflection coefficient over a suitable range of $\sin(u)$ as was discussed in Section IV, optimum values of α_n and γ_n were obtained. It was found that $N = 6$ terms were required to obtain reflection coefficients less than 0.01 everywhere except at the singular point $\sin(u) = 1$, which is the boundary between the propagating region and the evanescent region. Figures 2 and 3 show the reflection coefficients obtained in this manner as a function of $\sin(u)$. The large reflection coefficient for $\omega = 3$ at large values of $\sin(u)$ results from numerical dispersion. When these waves are important smaller values of D_x and D_y are required to describe them accurately in the interior and recover the desired boundary condition properties.

The data points shown in Figs. 2 and 3 are measured values of the magnitude of the reflection coefficient. They were obtained using the procedure outlined in Section IV. The points at $\sin(u) = 0.24, 0.49$, and 1.96 were obtained with a $p = 2$, $T = 30$ turn-on of the source, while the points at $\sin(u) = .982$ and 1.033 were obtained with a $p = 4$, $T = 500$ turn-on. Although errors due to the turn-on were larger in the neighborhood of $\sin(u) = 1$, the difficulties observed at 79° in Section IV were not obtained. Even with a $p = 2$, $T = 30$ turn-on the errors in the reflection coefficient were always less than its value. To obtain an accurate measure of the value of the reflection coefficient at a well-defined angle, however, a very slow turn-on was required.

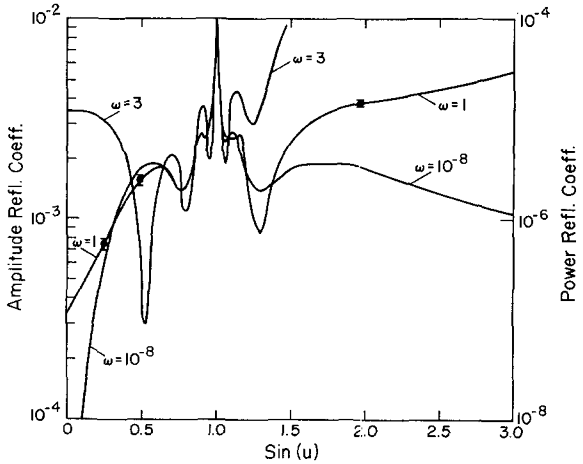

FIG. 2. The reflection coefficient vs sin(u) is presented for the boundary condition described in Eqs. (21)–(22) with $N = 6$. Sin(u) $= (c\tau/D_y)\sin(\frac{1}{2}k_yD_y)/\sin(\frac{1}{2}\omega\tau)$. The six values of α used were 0.5155, 0.2723, 0.1232, 0.05541, 0.02333, and 0.01030, and the corresponding values of γ were 1.6543, 0.4922, 0.09891, 0.01637, 0.002062, and 0.0001395 respectively. Other parameters were $c = 1$, $\tau = 0.0654$, and $D_x = D_y = 0.1$. Waves for sin(u) < 1 are propagating waves, while waves having sin(u) > 1 are evanescent. The data points are measured values of the reflection coefficient for $\omega = 1$.

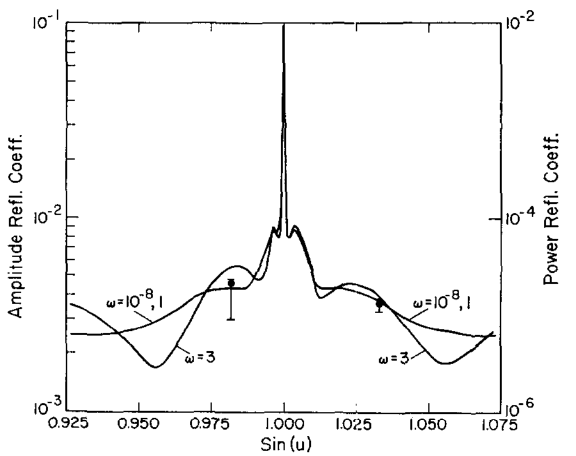

FIG. 3. The reflection coefficient in the neighborhood of the singular point sin(u) $= 1$ is presented for the boundary condition described in Fig. 2. The data points are measured values of the reflection coefficient for $\omega = 1$. The data point at sin(u) $= 1.033$ was obtained with $D_x = D_y = 0.095$ rather than $D_x = D_y = 0.1$ which was used for the other points in Figs. 2 and 3.

VI. CONCLUSION

Boundary conditions which act like an infinite region of free space in contact with the computational region can be constructed using projection operators. These operators use past data at the boundary which are processed in the form of the updating of three to six wave equations at the boundary. Reflection coefficients less than 1 % over a wide range of angles and frequencies are obtained; and evanescent waves can be handled with equivalent accuracy. At the same time externally specified waves may be launched into the computational region. The simplest procedure, which involves updating three wave equations at the boundary is nominally good for propagating waves only and for angles up to 89°. In practice, however, it is adequate for angles up to 60° or 70° when amplitude fluctuations lead to frequency shifts of $\delta\omega/\omega \lesssim 0.1$. A more complicated procedure involving the updating of six wave equations at the boundary handles both propagating and evanescent waves correctly and hence, does not suffer from the above restrictions.

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